Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 2, 1995

EFFECT OF DRY FRICTION ON THE SURFACE OF A GROWING SHEAR LAYER ON STRESS CONCENTRATION

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UDC 535.385

A prestressed elastic half-space whose surface was initially rigidly bonded to an undeformable base undergoes unloading due to a defect that originates at a point on the contact line and grows dynamically along the contact in the form of a shear layer. Dry friction occurs on the surface with a constant coefficient. Here, we obtain an analytic solution that determines the unknown stresses and the displacement at the boundary. Asymptotic solutions are found at the vertices of the layer. The effect of allowing for friction was examined in a similar problem (but not in an asymptotic formulation) in [1].

We will examine the plane deformation of an elastic half-space $y > 0$ in contact with a rigid half-space $y < 0$. We assign an initial uniform stress state in the elastic half-space

$$\sigma^0_{yy} = \sigma_0 < 0, \qquad \sigma^0_{xy} = \tau_0 > 0.$$

A defect is formed on the contact boundary at point $x = y = 0$ at the moment of time $t = 0$. This defect then propagates along the boundary to the right and left in the form of a shear layer. A shear stress is dynamically released on the layer, this stress being proportional to the normal stress:

$$\sigma_{xy}(t,x,0) = -k\sigma_{yy}(t,x,0) \qquad (k = \text{const}, \quad k > 0).$$

The stress field $\sigma_{ij}(t, x, y)$ is composed of the initial static and dynamic stresses $p_{ij}(t, x, y)$:

$$\sigma_{ij} = \sigma^0_{ij} + p_{ij}.$$

The following conditions must be satisfied for the sought dynamic state at the contact boundary

$$\begin{aligned} p_{xy}(t,x,0) &= -\tau_0 - k\sigma_0 - kp_{yy}(t,x,0) \quad (-v_1 t < x < v_2 t), \\ u(t,x,0) &= 0 \qquad (x < -v_1 t, \ x > v_2 t), \\ v(t,x,0) &= 0 \qquad (-\infty < x < \infty), \end{aligned} \tag{1}$$

where u and v are components of the displacement vector; the constants v_1 and v_2 are the subsonic velocities of the vertices of the layer.

Boundary-value problem (1) is closed by zero initial conditions. The displacements are reckoned from the values attained in the static state, which has no effect on boundary conditions (1). It should be noted that in the case of dry friction, two of the sought stresses are related only to one another, rather than being assigned explicitly on the layer. If $k = 0$, then boundary conditions (1) describe a shear layer without friction on the surface.

Employing the Laplace transform for time and the Fourier transform for the coordinate x (the parameters of the transforms being s and q), we obtain the following relation between the transforms of the displacements and the components of the stress vector at the boundary [2]:

Mining Institute, Siberian Division of the Russian Academy of Sciences, 630091 Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 166-172, March-April, 1995. Original article submitted April 20, 1994.

0021-8944/95/3602-0294$12.50

$$\mu u^{LF}(s,q,0) = S_{11}p_{xy}^{LF}(s,q,0) + S_{12}p_{yy}^{LF}(s,q,0),$$
$$\mu v^{LF}(s,q,0) = -S_{12}p_{xy}^{LF}(s,q,0) + S_{22}p_{yy}^{LF}(s,q,0),$$
$$S_{11} = -\frac{a_2^2 s^2 n_2}{R}, \quad S_{22} = -\frac{a_2^2 s^2 n_1}{R}, \quad S_{12} = -\frac{iq(n_2^2 + q^2 - 2n_1 n_2)}{R},$$
$$R = (n_2^2 + q^2)^2 - 4q^2 n_1 n_2, \quad n_i = \sqrt{q^2 + a_i^2 s^2} \qquad (i = 1,2). \tag{2}$$

Here, μ is the shear modulus; a_1 and a_2 are the inverses of the velocities of the longitudinal and transverse waves. We will seek the shear stresses in the form of a sum

$$p_{xy}(t,x,y) = -kp_{yy}(t,x,y) + \tau(t,x,y). \tag{3}$$

With this choice for p_{xy}, the boundary value of the new sought function τ on the layer is known and, as follows from (1), has the following form (the third argument of the functions at the boundary points will henceforth be omitted)

$$\tau(t,x) = -\tau_0 - k\sigma_0 \qquad (-v_1 t < x < v_2 t). \tag{4}$$

After substitution (3) with allowance for the condition $v = 0$ (which is satisfied over the entire boundary of contact of the half-spaces), Eqs. (2) take the form:

$$\mu s u^{LF}(s,q) = \frac{s}{iq} U\left(\frac{s}{iq}\right)\tau^{LF}(s,q), \qquad p_{yy}^{LF}(s,q) = P\left(\frac{s}{iq}\right)\tau^{LF}(s,q),$$
$$p_{xy}^{LF}(s,q) = T\left(\frac{s}{iq}\right)\tau^{LF}(s,q),$$
$$U\left(\frac{s}{iq}\right) = iq\frac{q^2 - n_1 n_2}{D}, \qquad P\left(\frac{s}{iq}\right) = iq\frac{n_2^2 + q^2 - 2n_1 n_2}{D},$$
$$T\left(\frac{s}{iq}\right) = \frac{a_2^2 s^2 n_1}{D}, \qquad D = ikq(n_2^2 + q^2 - 2n_1 n_2) + a_2^2 s^2 n_1. \tag{5}$$

One boundary condition $v = 0$ is satisfied in Eqs. (5). The conditions for the continuation of the layer with respect to the displacements u and for the layer itself with respect to τ should be accounted for in the first equation of (5), which connects the transforms of the indicated functions and is thus the equation used to determine the two functions u and τ at those points where they are unknown. If τ is found for the entire boundary, then the second and third equations of (5) can be used to obtain the stresses p_{yy} and p_{xy}.

The sought solution is self-similar, since the problem lacks characteristic units of measurement of length and time. For example, the solution could have the form $\tau(t, x) = x^n Q_0(x/t)$, where the exponent n is determined by the assigned boundary conditions. The method of functionally invariant solutions, employing the solution of the equations off the boundary as well, is usually used to solve self-similar mixed two-dimensional problems. However, we cannot directly use this method here, since we introduced the function τ. We will therefore use the method proposed in [2, 3]. This method allows us to find the solution of (5), i.e. the values u and τ at the boundary points, without information on the sought functions off the boundary.

The Laplace and Fourier transforms of such functions are homogeneous functions of the parameters of the transformations

$$\tau^{LF} = s^{-n-2}Q(s/q).$$

The homogeneous transformations are inverted using the formula (see [2, 4])

$$\tau(t,x)=\hat{\tau}_+(t,x)-\hat{\tau}_-(t,x),$$
$$\hat{\tau}(t,z)=-\frac{1}{2\pi i z n!}\int_0^t(t-\lambda)^n Q(iz/\lambda)\,d\lambda\qquad(z=x+i\xi). \tag{6}$$

Here, piecewise-analytic function $\hat{\tau}(t,z)$ is the analytic representation of the function $\tau(t,x)$ [5]; the signs $\pm$ denote the limiting values of this function at $z\to x\pm i0$ — the functions $\hat{\tau}_\pm=\hat{\tau}(t,x\pm i0)$.

Inverting transformations (5) in accordance with Eq. (6), we obtain an equation in the analytic representations of u and τ:

$$\mu\hat{u}^{(n+2)}(t,z)=\frac{z}{t}U(z/t)\hat{\tau}^{(n+1)}(t,z),$$
$$U(z/t)=\frac{1+n_1n_2}{k[2+2n_1n_2-(a_2z/t)^2]-(a_2z/t)^2n_1}, \tag{7}$$
$$n_i(z/t)=\sqrt{(a_iz/t)^2-1},\quad n_i>0\quad\text{at}\quad x/t>a_i^{-1},\quad \xi=0\,(i=1,2),$$

along with expressions for the remaining quantities at the boundary in terms of τ:

$$\hat{p}_{yy}^{(n+1)}(t,z)=P(z/t)\,\hat{\tau}^{(n+1)}(t,z),$$
$$\hat{p}_{xy}^{(n+1)}(t,z)=T(z/t)\,\hat{\tau}^{(n+1)}(t,z).$$

The superscripts in parentheses indicate the order of the derivative with respect to time. The single-valued branches of the radicals n_i in the plane z/t are marked by cuts on the real axis $(-a_i^{-1}, a_i^{-1})$. In this case, the values of the radicals are determined as:

$$n_i(x/t)=\begin{cases}\sqrt{(a_ix/t)^2-1}\ \mathrm{sgn}x & (|x/t|>a_i^{-1},\quad \xi=0),\\ \pm i\sqrt{1-(a_ix/t)^2} & (|x/t|<a_i^{-1},\quad \xi=\pm0).\end{cases}$$

The solution of (5) reduces to the solution of problem (7) for two piecewise-analytic functions. Meanwhile, we know from (1) and (4) that

$$u^{(n+2)}=\hat{u}_+^{(n+2)}-\hat{u}_-^{(n+2)}=0\qquad(x<-v_1t,\ x>v_2t),$$
$$\tau^{(n+1)}=\hat{\tau}_+^{(n+1)}-\hat{\tau}_-^{(n+1)}=(-\tau_0-k\sigma_0)^{(n+1)}\qquad(-v_1t<x<v_2t). \tag{8}$$

The function τ is constant on the layer, so we should set n = 0 everywhere in the formulas.

In order to find the solution of Eq. (7), we introduce the function

$$\Omega(t,z)=\frac{t}{z}w(z/t)\hat{u}^{\cdot\cdot}=w(z/t)U(z/t)\hat{\tau}^{\cdot}$$

(the points denote derivatives with respect to time).

The limiting values of the multiplier w satisfy the conditions

$$w_+-w_-=0\qquad(x<-v_1t,\ x>v_2t),$$
$$U_+w_+-U_-w_-=0\qquad(-v_1t<x<v_2t),$$
$$U_\pm=\frac{1-m_1m_2}{km(x/t)\mp i(a_2x/t)^2m_1},\qquad U_\pm\neq0\quad\text{at}\quad -v_1t<x<v_2t, \tag{9}$$
$$m(x/t)=2-2m_1m_2-(a_2x/t)^2,\qquad m_i(x/t)=\sqrt{1-(a_ix/t)^2}.$$

The radicals m_1 and m_2 are real-valued on the layer.

Equations (6) are Cauchy representations, so the functions $\hat{u}$ and $\hat{\tau}$ approach zero as $|z| \to \infty$, as does z^{-1}. Considering this condition to be satisfied for Ω, we can conclude that $w = O(z)$ at $|z| \to \infty$. We represent w as follows:

$$w(z/t) = (z/t + \alpha)w^0(z/t) \quad (w^0(z/t) \sim \text{const} \quad \text{at} \quad |z| \to \infty).$$

The constant α is still arbitrary. For the function w, we obtain a contact problem with a discontinuous coefficient G:

$$w_+^0 = Gw_-^0,$$

$$G(x/t) = \begin{cases} \dfrac{km(x/t) - i(a_2x/t)^2 m_1}{km(x/t) + i(a_2x/t)^2 m_1} & (-v_1 t < x < v_2 t), \\ 1 & (x < -v_1 t,\ x > v_2 t). \end{cases}$$

The solution of the given limit problem is the function

$$w^0(z/t) = \exp[-F(z/t)],$$

$$F(z/t) = \frac{1}{\pi} \int_{-v_1}^{v_2} \arctan\varphi(\eta) \frac{d\eta}{\eta - z/t}, \qquad \varphi(\eta) = a_2^2 m_1 \eta^2 / km(\eta).$$

Thus, the sought function for the problem of a shear layer with friction can be found with an arbitrary parameter

$$w(z/t) = (z/t + \alpha)\exp[-F(z/t)].$$

At $k = 0$, friction disappears from the edges and w takes the form [6]

$$w(z/t) = \sqrt{(z/t + v_1)(z/t - v_2)}.$$

Assuming that the solution of boundary-value problem (9) depends continuously on the parameter k, we find that $\alpha = -v_2$. The latter in turn means that

$$w(z/t) = (z/t - v_2)\exp[-F(z/t)].$$

The sign of k must be changed in boundary condition (1) when $\tau_0 < 0$. In this case, the requirement that the dependence on the friction parameter be continuous leads to the following solution of (9):

$$w(z/t) = (z/t + v_1)\exp[-F(z/t)] \qquad (k < 0).$$

The function w is constructed so that the discontinuity of Ω is known for the entire axis. In fact, considering (7) and (9), we obtain the following outside the layer

$$\Omega_+ - \Omega_- = \frac{t}{x} w(x/t) u^{\cdot\cdot}(t, x) = 0 \qquad (x < -v_1 t,\ x > v_2 t).$$

Similarly, on the layer we have

$$\Omega_+ - \Omega_- = w_+ U_+ (\hat{\tau}_+^{\cdot} - \hat{\tau}_-^{\cdot}) = w_+ U_+ \tau^{\cdot}(t, x) \qquad (-v_1 t < x < v_2 t).$$

Thus, the function Ω can be found from its jump on the real axis by using the Cauchy integral:

$$\Omega(t,z)=\frac{1}{2\pi i}\Big[\int\limits_{-v_1 t}^{v_2 t} w_+(\xi/t)U_+(\xi/t)\tau^{\cdot}(t,\xi)\frac{d\xi}{\xi-z}+\frac{A}{z+v_1 t}+\frac{B}{z-v_2 t}\Big],$$

$$w_+=\frac{(\xi/t-v_2)[km(\xi/t)-i(a_2\xi/t)^2 m_1]}{\sqrt{k^2m^2(\xi/t)+(a_2\xi/t)^4m_1^2}}\exp\Big[-\frac{1}{\pi}\text{v.p.}\int\limits_{-v_1}^{v_2}\arctan\varphi(\eta)\frac{d\eta}{\eta-\xi/t}\Big],$$

$$\mu\hat{u}^{\cdot\cdot}(t,z)=\frac{z}{t}w^{-1}(z/t)\Omega(t,z),\qquad \hat{\tau}^{\cdot}(t,z)=w^{-1}(z/t)U^{-1}\Omega(t,z), \tag{10}$$

$$\hat{p}^{\cdot}_{yy}(t,z)=\frac{2+2n_1n_2-(a_2z/t)^2}{(1+n_1n_2)w(z/t)}\Omega(t,z),$$

$$\hat{p}^{\cdot}_{xy}(t,z)=\frac{(a_2z/t)^2n_1}{(1+n_1n_2)w(z/t)}\Omega(t,z).$$

The functions U and m were determined in (9). We take the integral in the exponent to be the principal value.

The last two terms of Ω determine the boundary conditions $x=-v_1t$, $x=v_2t$ of the function at the alternation points. The constants A and B are found from boundary condition (8) for τ. Integrating $\hat{\tau}^{\cdot}$ in (10) over time, we obtain

$$\tau(t,x)=\tau(t,x)H(v_1t+x)+\int\limits_0^{-x/v_1}\Phi_1\Big(\frac{x}{\lambda}\Big)\frac{d\lambda}{\lambda}\qquad(-v_1t<x<0),$$

$$\tau(t,x)=\tau(t,x)H(v_2t-x)+\int\limits_0^{x/v_2}\Phi_2\Big(\frac{x}{\lambda}\Big)\frac{d\lambda}{\lambda}\qquad(0<x<v_2t), \tag{11}$$

where H(...) is the Heaviside function; Φ_1 and Φ_2 are the jumps of the function $\Phi(z/t)=t\Omega(t,z)/[w(z/t)U(z/t)]$ on the intervals $(-v_1t<x<0)$ and $(0<x<v_2t)$.

Satisfaction of boundary condition (8) for τ requires that the integrals in (11) be equated to zero:

$$\int\limits_0^{1/v_1}\Phi_1\Big(-\frac{1}{\eta}\Big)\frac{d\eta}{\eta}=0,\qquad \int\limits_0^{1/v_2}\Phi_2\Big(\frac{1}{\eta}\Big)\frac{d\eta}{\eta}=0. \tag{12}$$

These equalities make it possible to find the constants A and B.

In the problem being examined, $\tau^{\cdot}=0$ on the layer. Thus,

$$\Omega(t,z)=\frac{1}{2\pi i}\Big(\frac{A}{z+v_1t}+\frac{B}{z-v_2t}\Big).$$

In calculations of A and B, it is expedient to transform Eqs. (12) so as to eliminate integration over the neighborhood of singular points v_1^{-1} and v_2^{-1}. To do this, before calculating the jump of $\hat{\tau}$ on the layer in (11), we should make a substitution of variables to change over from integration over t to integration over z/t. We then deform the contours of integration over the edges in the upper and lower half-planes. As a result, instead of (12) we have conditions in which the integrals are taken on the layer over a segment without singular points and along a straight line, such as $z/t=(1/2)(v_2-v_1)+iy/t$, that lies in the complex plane z/t parallel to the imaginary axis:

$$Av_2-Bv_1=0,\qquad A+B=\pi(\tau_0+k\sigma_0)/J,$$

$$J=\text{Re}\int\limits_0^\infty\frac{d\xi}{U(\gamma+i\xi)w(\gamma+i\xi)(v_1+\gamma+i\xi)(v_2-\gamma-i\xi)},\qquad \gamma=\frac{v_2-v_1}{2}.$$

The stresses on the continuation of the layer are calculated from the formulas

$$p_{yy}(t,x) = C\,\mathrm{Re}\int\limits_{x/t+i0}^{\infty} \frac{2+2n_1n_2-a_2^2u^2}{i(1+n_1n_2)w(u)(v_1+u)(u-v_2)}\,du,$$

$$p_{xy}(t,x) = C\,\mathrm{Re}\int\limits_{x/t+i0}^{\infty} \frac{a_2^2u^2n_1}{i(1+n_1n_2)w(u)(v_1+u)(u-v_2)}\,du,$$

$$C = \frac{(A+B)}{\pi} = \frac{(\tau_0+k\sigma_0)}{J}.$$

For the points on the layer, it is convenient to use other representations obtained by transformation of the path of integration in exactly the same manner as was done in the calculation of the constants A and B:

$$p_{yy}(t,x) = C\,\mathrm{Re}\int\limits_{0}^{\infty} \frac{2+2n_1(u_\gamma)n_2(u_\gamma)-a_2^2u_\gamma^2}{[1+n_1(u_\gamma)n_2(u_\gamma)]w(u_\gamma)(v_1+u_\gamma)(v_2-u_\gamma)}\,d\xi -$$

$$-C\,\mathrm{Re}\int\limits_{\gamma+i0}^{x/t+i0} \frac{2-2m_1m_2-a_2^2u^2}{i(1-m_1m_2)w_+(u)(v_1+u)(u-v_2)}\,du,$$

$$p_{xy} = -kp_{yy} - \tau_0 - k\sigma_0, \quad u_\gamma = \gamma + i\xi.$$

We use the above to obtain the following stress asymptotes in the neighborhood of the vertices:

$$p_{yy} \sim -C\frac{F(v_2)(v_2-x/t)^{f(v_2)/\pi-1}}{(v_1+v_2)^{1+f(-v_1)/\pi}(1-f(v_2)/\pi)} \qquad (x \to v_2t-0);$$

$$p_{yy} \sim -C\frac{\pi F(-v_1)(v_1+x/t)^{-f(-v_1)/\pi}}{(v_1+v_2)^{2-f(v_2)/\pi}f(-v_1)} \qquad (x \to -v_1t+0);$$

$$F(v) = \frac{a_2^2v^2m_1(v)}{\sqrt{k^2m^2(v)+a_2^4v^4m_1^2(v)}}\exp\Big[\frac{1}{\pi}\int\limits_{-v_1}^{v_2}\ln|\eta-v|f'(\eta)\,d\eta\Big],$$

$$f(\eta) = \arctan\varphi(\eta).$$

The asymptotes obtained here coincide with the those obtained in [1] if one of the media is regarded as nondeformable.

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